Fuzzy predicate values

Suppose we have a population of N individuals, numbered $1..N$. A function, $H(i)$, yields a real number indicating a certain feature, such as "height", of each individual.

We wish to define a fuzzy predicate, $H_W(i)$, which yields a real number indicating the degree of membership in a fuzzy set, such as "the set of tall individuals". The fuzzy predicate will yield values near 0.0 for individuals that are clearly not tall, values near 1.0 for individuals that are clearly tall, and values between 0.0 and 1.0 for individuals that are between not tall and tall. $H_W(i)$ interpolates the relevant range of $H(i)$ into the range $[0.0 \dots 1.0]$, as follows:

$$H_W(i) = \frac{H(i) - H_F}{H_T - H_F}$$

When a feature is evaluated across the entire fuzzy set Hw, each individual is weighted according to its membership in the set. For example, to measure the average mass, M, across all members of H_W, each member is weighted by $H_W(i)$, as follows:

$$M(H_W) = \frac{2}{N} \sum_{i=1}^{N} M(i) H_W(i)$$

Note that the factor of 2 indicates that half of instances are included as members and half are excluded.

We wish to use fuzzy predicates such as $H_W(i)$ to express logical relationships. In order to approximate conventional logical predicates, our fuzzy predicates need to exhibit a few mathematical properties. We need at least:

1. fuzzy truth values, H_F and H_T, to represent "false" and "true",
2. a conjunction operator to predict coincidence of "true" predicates,

3. a negation operator to swap between "true" and "false".

Multiplication will be used to calculate conjunction, as it is with Bayesian probabilities and likelihoods, where 0.0 represents "false". Numeric negation, $1.0 - H_w$, will be used to calculate logical negation. To make conjunction and negation work, all fuzzy set non-members should have $H(i)$ near to H_F, and likewise all fuzzy set members should have $H(i)$ near to H_T. This can be arranged by choosing H_F to be the mean of $H(i)$ across fuzzy set non-members, and H_T to be the mean of $H(i)$ across fuzzy set members. Choosing H_F and H_T in this way ensures that non-members have zero net influence on a feature evaluated across members, while members have zero net influence on a feature evaluated across non-members. It also ensures that the member and non-member sets are complementary sets, where each individual is included with total weight 1.0 in the union of the fuzzy set and its complement.

It's a little challenging to calculate H_F and H_T, since they are derived from set membership and they also determine set membership. The proof below shows that H_F and H_T are exactly 1 standard deviation from the mean of $H(i)$.

$$H_T = \frac{2}{N}\sum_{i=1}^{N} H(i)Hw(i)$$

$$H_T = \frac{2}{N}\sum_{i=1}^{N} H(i)\frac{H(i)-H_F}{H_T - H_F}$$

Assuming that the mean of $H(i)$ is zero, and $H_F = -H_T$,

$$H_T = \frac{2}{N}\sum_{i=1}^{N} H(i)\frac{H(i)+H_T}{2H_T}$$

$$H_T = \frac{1}{2H_T}\frac{2}{N}\sum_{i=1}^{N} H(i)(H(i) + H_T)$$

$$H_T^2 = \frac{1}{N}\sum_{i=1}^{N} H(i)(H(i) + H_T)$$

$$H_T^2 = \frac{1}{N}\sum_{i=1}^{N} H(i)^2 + H(i)H_T$$

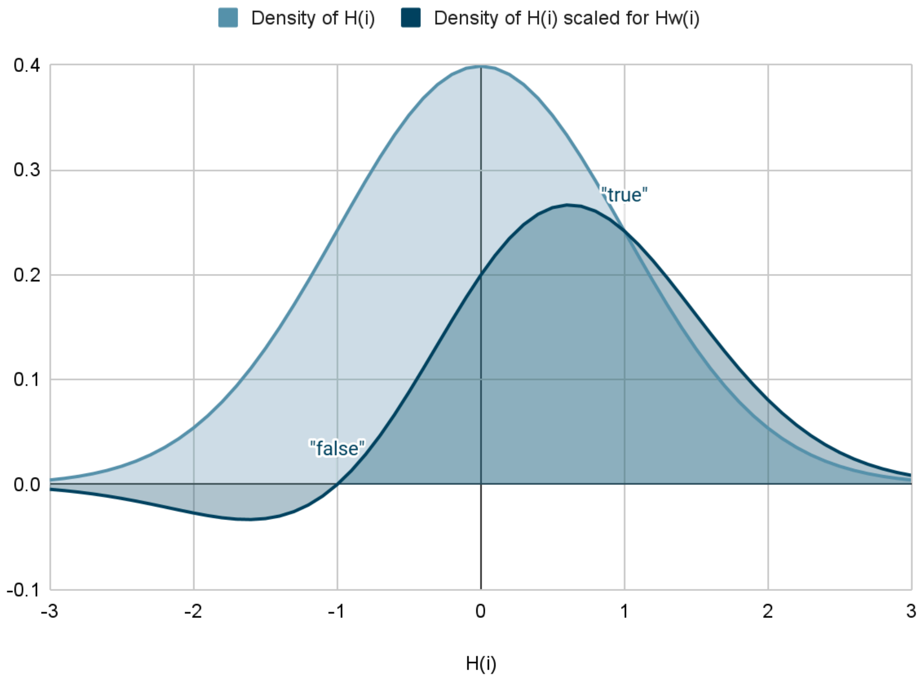
$$H_T^2 = \frac{1}{N}\sum_{i=1}^{N} H(i)^2 + \frac{1}{N}\sum_{i=1}^{N} H(i)H_T$$

$$H_T^2 = \frac{1}{N}\sum_{i=1}^{N} H(i)^2 + 0H_T$$

$$H_T^2 = \frac{1}{N}\sum_{i=1}^{N} H(i)^2$$

$$H_T = \sqrt{\frac{1}{N}\sum_{i=1}^{N} H(i)^2}$$

The chart below shows the density of individuals with each predicate value $H(i)$, and the same density scaled according to membership in the fuzzy set $H_W(i)$. For clarity, the individuals follow a normal distribution with standard deviation 1.0. The data points at $H(i) = -1$ and $H(i) = 1$ are labeled as "false" and "true". These are assigned membership 0.0 and 1.0 respectively in the fuzzy set $H_W(i)$. $H(i) = 1$ is the average of the scaled density below.



Notice that when H_F and H_T are centered in this way, many individuals are assigned membership below 0.0 or above 1.0. These outliers can be understood as anti-members and super-members of the fuzzy set respectively. Anti-members have characteristics opposite to those of set members, while super-members have characteristics exaggerating those of set members. After negation, H_T would get weight 0.0, and H_F would get weight 1.0, and super-members of the fuzzy set would become anti-members of its complement.

These fuzzy predicate values do the job of probabilities in Bayesian inference. They define a generalization of probability that extends below 0.0 and above 1.0. These fuzzy predicate values can be regarded as the net rate of an event relative to the baseline rate. Using these fuzzy predicate values, it is possible to infer Bayesian likelihoods that express negative influences from hypotheses that are contrary to the correct hypotheses. A negative likelihood indicates that a hypothesis is not only unlikely to have been the cause of the observed event, but is in fact more likely to have prevented the observed event.